Hence $a+b=49$, $a-b=7$, $a=28$, $b=21$. The geometric verification is evident.

Also solved by B. F. Finkel, A.M., M.Sc., 204 St. Marks Square, Philadelphia, Pa.; J. K. Ellwood, Principal Colfax Sub-District School, Pittsburg, Pa.; A. H. Holmes, Brunswick, Me.; and G. B. M. Zerr, A. M., Ph. D., Parsons, W. Va.; L. E. Newcomb, Los Gatos, California; G. I. Hopkins, Warren High School, Warren, Pa.; J. Scheffer, Hagerstown, Md.

217. Proposed by G. W. DRAKE, Fayetteville, Ark.

If one of the principal axes of a cone which stands on a given base be always parallel to a given right line, the locus of the vertex is an equilateral hyperbola or a right line according as the base is a central conic or a parabola. [Exercise 40, page 94, C. Smith's *Solid Geometry*].

Solution by WILLIAM HOOVER, Ph. D., Professor of Mathematics in the Ohio State University, Athens, O.

Let the given central conic be

$$x^2/a+y^2/b=1,\ z=0 \qquad\text{.......(1)},$$

and regarded as the central conicoid

$$x^2/a+y^2/b+z^2/c=1 \qquad\text{.......(2)},$$

in which the semi-axis c vanishes.

This solution will employ the principle that the principal axes of a cone, which envelope a given conicoid, are normals to the three confocals which pass through the vertex. The confocals to (2) are given by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=1 \qquad\text{.......(3)},$$

and so the confocals to (1), by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{\lambda}=1 \qquad\text{.......(4)}.$$

If $(\xi,\ \eta,\ \varsigma)$ be the vertex of the cone, a normal to one of the system (4) is

$$\frac{[a+\lambda][x-\xi]}{\xi}+\frac{[b+\lambda][y-\eta]}{\eta}+\frac{[c+\lambda]\ [z-\varsigma]}{\varsigma}=1 \qquad\text{.......(5)},$$

and if l, m, n be the direction-cosines of the given line, and ρ an undetermined number, we should have, for the normal to (4) at $(\eta,\ \xi,\ \varsigma)$,

$$l=\frac{\rho\xi}{a+\lambda},\quad m=\frac{\rho\eta}{b+\lambda},\quad n=\frac{\rho\varsigma}{\lambda} \qquad\text{.......(6)},$$

$$\text{giving } a+\lambda=\frac{\rho\xi}{l},\quad b+\lambda=\frac{\rho\eta}{m},\quad \lambda=\frac{\rho\varsigma}{n} \qquad\text{.......(7)},$$

$$\text{and } l\xi+m\eta+n\varsigma=\rho\left(\frac{\xi^2}{a+\lambda}+\frac{\eta^2}{b+\lambda}+\frac{\xi^2}{\lambda}\right)=\rho \qquad\text{.......(8)},$$

since (ξ, η, ς) is on (4). From the first two of (7),

$$a-b=\rho[\xi/l-\eta/m] \qquad (9),$$

and from the first and third of (7), $a=\rho[\xi/l-\varsigma/n]$ (10).

(8) and (9) give $\quad a-b=[l\xi+m\eta+n\varsigma][\xi/l-\eta/m]$ (11),

and (9) and (10) give $[a-b][\xi/l-\varsigma/n]=a[\xi/l-\eta/m]$ (12),

showing that the locus of the vertex, when the base is a central conic (1), is the intersection of the hyperboloid (11) by the plane (12), or an equilateral hyperbola.

For the second part, let the base be an indefinitely thin elliptic or hyperbolic paraboloid given by

$$x^2/a+y^2/b=2z \qquad (13),$$

and the paraboloids confocal to (13), by

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}=2z+\lambda \qquad (14).$$

The normals to (14) through (ξ, η, ς) are given by

$$\frac{[a+\lambda][x-\xi]}{\xi}+\frac{[b+\lambda][y-\eta]}{\eta}=-\frac{z-\varsigma}{-1} \qquad (15),$$

and we have again

$$l=\frac{\rho\xi}{a+\lambda}, \quad m=\frac{\rho\eta}{b+\lambda}, \quad n=-\rho \qquad (16),$$

giving $\quad a+\lambda=\rho\xi/l, \quad b+\lambda=\rho\eta/m$ (17).

These give $\quad l\xi+m\eta+2n\varsigma=\rho\left(\frac{\xi^2}{a+\lambda}+\frac{\eta^2}{b+\lambda}-2\varsigma\right)=\rho\lambda=-n\lambda$ (18).

This gives $\quad -\lambda=\dfrac{l\xi+m\eta+2n\varsigma}{n}$ (19).

Adding (19) to the first of (17) and substituting from (16),

$$[l^2-n^2]\xi+lm\eta+2ln\varsigma-aln=0 \qquad (20).$$

Eliminating λ and ρ from (16) and (17),

$$a-b=-n[\xi/l-\eta/m] \qquad (21).$$

Therefore in this case the locus of the vertex is a straight line, the intersection of the planes (20) and (21).